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IDEAL THEORY IN f-ALGEBRAS

BY

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ABSTRACT. The paper deals mainly with the theory of algebra ideals and order ideals in f-algebras. Necessary and sufficient conditions are established for an algebra ideal to be prime, semiprime or idempotent. In a uniformly complete f-algebra with unit element every algebra ideal is an order ideal iff the f-algebra is normal. This result is based on the fact that the range of every orthomorphism in a uniformly complete normal Riesz space is an order ideal.

1. Introduction. In the present paper we investigate how far several well-known results about ideal theory in the space $C(X)$ of all real continuous functions on a completely regular Hausdorff space X can be extended to the most natural generalization of $C(X)$, a uniformly complete Archimedean f-algebra with unit element. It should be observed that the class of these f-algebras contains the class of all $C(X)$ as a proper subset. As an example we mention the space $\mathfrak{M}([0, 1])$ of all real Lebesgue measurable functions on $[0, 1]$ with the usual identification of almost equal functions. This is a uniformly complete Archimedean f-algebra with unit element (with respect to the familiar operations and the natural partial ordering). As is well known (see e.g. [**18**, Example 5.A]), the only positive linear functional on $\mathfrak{M}([0, 1])$ is the null functional. From this it follows that $\mathfrak{M}([0, 1])$ cannot be f-algebra-isomorphic to any $C(X)$, since every point evaluation in $C(X)$ is a nontrivial positive linear functional.

Abstract f-ring theory and f-algebra theory have been studied by many authors (see e.g. [**1, 5, 6, 7, 13, 14, 17, 21**]). Some of these authors (see e.g. §8 in the paper [**6**] by G. Birkhoff and R. S. Pierce) define a f-ring as a lattice ordered ring with the property that $u \wedge v = 0$, $w \geqslant 0$ implies $(uw) \wedge v = (wu) \wedge v = 0$. Others (see e.g. Definition 9.1.1 in the book [**5**] by A. Bigard, K. Keimel and S. Wolfenstein, and Chapter IX, §2 in the book [**7**] by L. Fuchs) define an f-ring as a lattice ordered ring which is isomorphic to a subdirect union of totally ordered rings. It is often desirable to have the equivalence of the two definitions available. However, any known equivalence proof is based on arguments using Zorn's lemma. If one uses the second definition, it is possible to prove a certain number of standard theorems on f-rings by means of the "metamathematical" observation that any identity holding in every totally ordered ring holds in every f-ring.

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Another point to note is that in the literature many results on (uniformly complete) Archimedean f-algebras with unit element are proved by using representation theorems. However, all representation theorems of this type depend heavily on Zorn's lemma.

We adopt in this paper the original Birkhoff-Pierce definition of an f-algebra (i.e., a lattice ordered algebra with the additional property that $u \wedge v = 0$, $w \geqslant 0$ implies $(uw) \wedge v = (wu) \wedge v = 0$) as our starting point. All our proofs will be intrinsic (i.e., purely algebraic and lattice theoretical proofs in the f-algebra itself). In other words, we shall not make use of any representation theorem or metamathematical theorem of the kind mentioned above. We believe that one of the advantages of this approach is that Zorn's lemma is not used unnecessarily.

In §3 we present a review of elementary f-algebra theory. We believe that some of the results are new. For several of the known results the proofs are new. The main subject of §4 is the theory of semiprime and idempotent algebra ideals in f-algebras. Several results of M. Henriksen (see **[12]**) are extended. In §5 it is proved that in a uniformly complete Archimedean normal Riesz space the range of every orthomorphism is an order ideal, a considerable improvement of the result so far known. This is applied in §6 to show that a uniformly complete Archimedean unitary f-algebra A is normal iff every algebra ideal in A is an l-ideal.

For the terminology and notations used and for the general theory of Riesz spaces we refer to **[20]**. We wish to express our gratitude to A. C. Zaanen for his valuable suggestions.

2. Preliminaries. Let L be a Riesz space (vector lattice). For any $f \in L$ we denote $f^+ = f \vee 0$, $f^- = (-f) \vee 0$ and $|f| = f \vee (-f)$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. The elements $f, g \in L$ are called disjoint if $|f| \wedge |g| = 0$, which is denoted by $f \perp g$. The disjoint complement of a subset E of L is denoted by E^d, i.e., $E^d = \{f \in L: f \perp g \text{ for all } g \in E\}$. A linear subspace I of L is called an order ideal (o-ideal) whenever it follows from $|g| \leqslant |f|, f \in I$ that $g \in I$. An o-ideal is a Riesz subspace, i.e. $f, g \in I$ implies $f \vee g \in I$ and $f \wedge g \in I$. The principal o-ideal generated by $f \in L$ is denoted by I_f. The o-ideal P in L is called a prime o-ideal whenever $f \wedge g \in P$ implies $f \in P$ or $g \in P$ (equivalently, $f \wedge g = 0$ implies $f \in P$ or $g \in P$). The o-ideal B in L is called a band whenever it follows from $f = \sup f_\tau$, $f_\tau \in B$ for all τ, that $f \in B$. Disjoint complements are bands, and L is Archimedean iff every band in L is a disjoint complement (see **[20**, Theorem 22.3]).

Given the element $v \geqslant 0$ in L, the sequence $\{f_n: n = 1, 2, \dots\}$ is said to converge v-uniformly to the element $f \in L$ whenever for every $\varepsilon > 0$ there exists a natural number N_ε such that $|f - f_n| \leqslant \varepsilon v$ holds for all $n \geqslant N_\varepsilon$. We denote this by $f_n \to f(v)$. The sequence $\{f_n: n = 1, 2, \dots\}$ is said to converge (relatively) uniformly to $f \in L$ if $f_n \to f(v)$ for some $0 \leqslant v \in L$. We denote this by $f_n \to f$ (r.u.). The notions of v-uniform Cauchy sequence and of uniform Cauchy sequence are defined likewise. The Riesz space L is Archimedean iff every uniformly convergent sequence in L has a unique limit (see **[20**, Theorem 63.2]). If $\{f_n: n = 1, 2, \dots\}$ is a v-uniform Cauchy sequence in the Archimedean Riesz space L and $f_n \to f(w)$, then also $f_n \to f(v)$. The Riesz space L is called v-uniformly complete whenever every v-uniform Cauchy sequence in L has a unique limit, and L is called uniformly complete whenever L is v-uniformly complete for all $0 \leqslant v \in L$. Note that it follows from our definition of uniform completeness that a uniformly complete Riesz space is Archimedean.

If $D \subset L$, we define the pseudoclosure D' of D to be the set of all $f \in L$ for which there exist $f_n \in D$ $(n = 1, 2, \ldots)$ with $f_n \to f$ (r.u.). The subset D of L is called uniformly closed whenever $D = D'$. The uniformly closed sets are the closed sets for a topology in L, the so-called uniform topology. The closure in this topology of a subset D of L is denoted by D^-. Obviously $D' \subset D^-$, but in general $D' \neq D^-$. For details we refer to §16 of [**20**]. If I is an o-ideal in L, then I^- is likewise an o-ideal (see [**20**, Theorem 63.1]).

The Riesz space L is called normal if $u \wedge v = 0$ in L implies that $L = \{u\}^d + \{v\}^d$. Equivalent conditions are

(i) $L = \{f^+\}^d + \{f^-\}^d$ for all $f \in L$.

(ii) $\{u_1 \wedge \cdots \wedge u_n\}^d = \{u_1\}^d + \cdots + \{u_n\}^d$ for all $0 \leqslant u_1, \ldots, u_n \in L$ and all n.

(iii) If $u_1 \wedge \cdots \wedge u_n = 0$, $n \in \mathbf{N}$, then $L = \{u_1\}^d + \cdots + \{u_n\}^d$.

We omit the easy proof. For other conditions equivalent to normality we refer to [**15**, Theorem 9]. Note that the quasiprincipal projection property (i.e. $L = \{u\}^d \oplus \{u\}^{dd}$ for all $0 \leqslant u \in L$) implies normality.

The Riesz algebra (i.e. lattice ordered algebra) A is said to be an f-algebra whenever $u \wedge v = 0$, $0 \leqslant w \in A$ implies $(uw) \wedge v = (wu) \wedge v = 0$ (see [**6**]). Any Archimedean f-algebra is commutative (see [**3**] for a proof not using representation theorems). We list some simple properties of f-algebras.

(i) Multiplication by a positive element is a Riesz homomorphsm, i.e., for $0 \leqslant u \in A$ and $f, g \in A$ we have $u(f \wedge g) = (uf) \wedge (ug)$, $u(f \vee g) = (uf) \vee (ug)$, $(f \wedge g)u = (fu) \wedge (gu)$ and $(f \vee g)u = (fu) \vee (gu)$.

(ii) $|fg| = |f| \cdot |g|$ for all $f, g \in A$.

(iii) $f \perp g$ implies $fg = 0$.

(iv) $ff^+ \geqslant 0$ and $f^2 \geqslant 0$ for all $f \in A$.

For the sake of completeness we will give the proofs of these properties.

Proof. (i) Take $f, g \in A$ and $0 \leqslant u \in A$. Then $(f - f \wedge g) \wedge (g - f \wedge g) = 0$, so $\{uf - u(f \wedge g)\} \wedge \{ug - u(f \wedge g)\} = 0$, which implies that $(uf) \wedge (ug) = u(f \wedge g)$. Now it follows from $f \vee g = f + g - f \wedge g$ that $u(f \vee g) = uf + ug - (uf) \wedge (ug) = (uf) \vee (ug)$. The right multiplication with u is treated analogously.

(ii) If $f, g \in A$, then it follows directly from the definition of an f-algebra that $f^+g^+ \perp f^+g^-$, $f^+g^+ \perp f^-g^+$, $f^-g^- \perp f^+g^-$ and $f^-g^- \perp f^-g^+$; hence $f^+g^+ + f^-g^- \perp f^+g^- + f^-g^+$. Using [**20**, Theorem 11.10(ii)], it follows from $fg = (f^+g^+ + f^-g^-) - (f^+g^- + f^-g^+)$ that $(fg)^+ = f^+g^+ + f^-g^-$ and $(fg)^- = f^+g^- + f^-g^+$ and therefore $|fg| = (fg)^+ + (fg)^- = (f^+ + f^-)(g^+ + g^-) = |f| \cdot |g|$.

(iii) Suppose $f, g \in A$ such that $f \perp g$, i.e., $|f| \wedge |g| = 0$. Then $(|f| \cdot |g|) \wedge |g| = 0$, so $(|f| \cdot |g|) \wedge (|f| \cdot |g|) = 0$; hence $|fg| = |f| \cdot |g| = 0$, i.e., $fg = 0$.

(iv) For any $f \in A$ we have $ff^+ = (f^+)^2 - f^-f^+ = (f^+)^2 \geq 0$, and $f^2 = (f^+)^2 + (f^-)^2 \geq 0$.

If A is a semiprime f-algebra (i.e. $f^n = 0$ implies $f = 0$), then $f \perp g$ iff $fg = 0$ (indeed, if $fg = 0$, then $(|f| \wedge |g|)^2 \leq |f| \cdot |g| = |fg| = 0$ implies that $(|f| \wedge |g|)^2 = 0$, so $|f| \wedge |g| = 0$) and so we have for all subsets E of A that $E^d = \mathrm{ann}(E)$, with $\mathrm{ann}(E) = \{f \in A : fg = 0 \text{ for all } g \in E\}$.

By an r-ideal I in A we always mean an algebra ideal (i.e., a linear subspace which is also a ring ideal). The r-ideal generated by $f_1, \ldots, f_n \in A$ is denoted by $(f_1, \ldots, f_n)$. An r-ideal in A which is an o-ideal as well is called an l-ideal.

We remind the reader that the order bounded linear mapping π from an Archimedean Riesz space L into itself is called an orthomorphism if $f \perp g$ in L implies that $\pi f \perp g$ (equivalently, $\pi(B) \subset B$ for all bands B in L). The collection $\mathrm{Orth}(L)$ of all orthomorphisms on L is an Archimedean Riesz space with $(\pi_1 \vee \pi_2)u = (\pi_1 u) \vee (\pi_2 u)$ and $(\pi_1 \wedge \pi_2)u = (\pi_1 u) \wedge (\pi_2 u)$ for all $u \geq 0$ (see [**19**, Theorem 4.10]). In particular, every orthomorphism is the difference of two positive orthomorphisms, which implies that the above definition of orthomorphism is equivalent to the one given in [**5**, Chapter 12]. Moreover, $\mathrm{Orth}(L)$ is an Archimedean (hence commutative) f-algebra with unit element, with composition as multiplication. Every orthomorphism is order continuous and its kernel is a band (see [**19**, Theorem 4.9 and Corollary 4.11]). It is not difficult to prove that $\mathrm{Orth}(L)$ is uniformly complete whenever L is uniformly complete.

In an Archimedean f-algebra A for any $f \in A$ we denote by π_f the orthomorphism defined by $\pi_f g = fg$ for all $g \in A$. The mapping $f \to \pi_f$ is a Riesz and algebra homomorphism of A onto the set $\hat{A} = \{\pi_f : f \in A\}$, which is a Riesz subspace and r-ideal in $\mathrm{Orth}(A)$. The mapping $f \to \pi_f$ is injective iff A is semiprime, so in this case A and $\hat{A}$ are f-algebra-isomorphic (see [**5**, Théorème 12.3.8]). Finally, note that A is unitary iff $\hat{A} = \mathrm{Orth}(A)$.

3. Elementary properties of f-algebras.

PROPOSITION 3.1. *Let A be an Archimedean f-algebra.*
(i) *$rf \in I_f'$ for all $r, f \in A$.*
(ii) *Any uniformly closed o-ideal I is an l-ideal.*

PROOF. (i) We may assume that $r, f \geq 0$. We assert that

$$(1) \qquad 0 \leq rf - rf \wedge nf \leq n^{-1} fr^2, \qquad n = 1, 2, \ldots.$$

Indeed, it follows from $(nf - rf \wedge nf) \wedge (rf - rf \wedge nf) = 0$ that

$$\{rf - r(n^{-1}rf \wedge f)\} \wedge \{rf - rf \wedge nf\} = 0,$$

and hence $rf = \{r(n^{-1}rf \wedge f)\} \vee (rf \wedge nf)$, so

$$0 \leq rf - rf \wedge nf \leq r(n^{-1}rf \wedge f) \leq n^{-1}fr^2.$$

This shows that $rf \wedge nf \uparrow rf(fr^2)$. Since $rf \wedge nf \in I_f$, it follows that $rf \in I_f'$.

(ii) If $f \in I$, then $rf \in I_f' \subset I$ for all $r \in A$.

PROPOSITION 3.2. *In an Archimedean f-algebra A with unit element e the following statements hold.*

(i) $u \wedge ne \uparrow u(u^2)$ *for every* $0 \leqslant u \in A$, *i.e.*, $I'_e = A$ (*in particular e is a weak unit*).

(ii) $I_f^- = I_{f^2}^-$ *for every* $f \in A$.

(iii) A *is semiprime.*

PROOF. (i) Follows from inequality (1) above.

(ii) We may assume that $f \geqslant 0$. By Proposition 3.1, $f^2 \in I_f^-$. Conversely, $(e - nf)(e - nf)^+ \geqslant 0$ implies that $(f - nf^2)^+ \leqslant n^{-1}e$, $n = 1, 2, \ldots$, and hence $f \wedge nf^2 \uparrow f(e)$. This shows that $f \in I_{f^2}^-$, and we are done.

(iii) If $f^2 = 0$, then by (ii), $I_f^- = \{0\}$, i.e., $f = 0$.

The inequality $0 \leqslant u - u \wedge ne \leqslant n^{-1}u^2$ $(n = 1, 2, \ldots)$ from Proposition 3.2(i) can be generalized in the following way.

LEMMA 3.3. *Let A be an Archimedean f-algebra with unit element e and* $0 \leqslant u \in A$. *Then*

$$0 \leqslant u \wedge me - u \wedge ne \leqslant n^{-1}(u \wedge me)(u \wedge ne) \quad \textit{for all } m \geqslant n.$$

PROOF. Put $u_n = u \wedge ne$ $(n = 1, 2, \ldots)$. For $m \geqslant n$ we have $(u_m - u_n) \wedge (ne - u_n) = 0$. Multiplying the second element by $n^{-1}u_m$ we get $(u_n - u_m) \wedge (u_m - n^{-1}u_m u_n) = 0$, and hence $u_m = u_n \vee (n^{-1}u_m u_n)$. It follows that $u_m - u_n = (n^{-1}u_m u_n - u_n)^+ \leqslant n^{-1}u_m u_n$.

In [**14**, Theorems 3.3 and 3.8], M. Henriksen and D. G. Johnson prove by means of representation that in any Archimedean f-algebra A with unit element e which is e-uniformly complete, every $u \geqslant e$ has an inverse and every $u \geqslant 0$ has a square root (i.e., there exists $0 \leqslant v \in A$ such that $u = v^2$). We shall indicate elementary proofs of these facts.

THEOREM 3.4. *Let A be an Archimedean f-algebra with unit element e which is e-uniformly complete. If* $0 < v \leqslant u$ *and* v^{-1} *exists in* A, *then* u^{-1} *exists in* A.

PROOF. We may assume that $v = e$.

(1) Suppose $e \leqslant u \leqslant \alpha e$ for some $\alpha > 1$. Since $0 \leqslant e - \alpha^{-1}u \leqslant (1 - \alpha^{-1})e$, the series $\sum_{n=0}^{\infty}(e - \alpha^{-1}u)^n$ converges e-uniformly and its sum is equal to αu^{-1}, so u^{-1} exists. Note that in this case the construction of the inverse is the same as in Banach algebras.

(2) Suppose $e \leqslant u$ and put $u_n = u \wedge ne$ $(n = 1, 2, \ldots)$. From (1) it follows that u_n^{-1} exists for all n. Using Lemma 3.3 we have for $m \geqslant n$

$$0 \leqslant u_n^{-1} - u_m^{-1} = u_n^{-1}u_m^{-1}(u_m - u_n) \leqslant u_n^{-1}u_m^{-1}\left(n^{-1}u_n u_m\right) = n^{-1}e,$$

so $\{u_n^{-1}\colon n = 1, 2, \ldots\}$ is an e-uniform Cauchy sequence. From the hypothesis it follows that there exists $0 \leqslant w \in A$ such that $u_n^{-1} \to w(e)$. Combining this with $0 \leqslant u_n \uparrow u(u^2)$ we infer $w = u^{-1}$.

In the next example we will show that the condition of e-uniform completeness cannot be dropped in the last theorem.

EXAMPLE 3.5. Let A be the f-algebra consisting of all $f \in C([0, 1])$ which are piecewise polynomials (finitely many pieces). Then A has a unit element e ($e(x) = 1$ for all $x \in [0, 1]$), but A is not e-uniformly complete. If we denote by i the function $i(x) = x$, then $i + e$ has no inverse in A.

PROPOSITION 3.6 (FOR THE $C(X)$ CASE, [9, LEMMA 1.5]). *Let A be an Archimedean f-algebra with unit element e which is e-uniformly complete. If $0 \leqslant u \in A$, then $u = vw$ with $0 \leqslant v \leqslant e$ and w^{-1} exists in A.*

PROOF. Observe first that in any f-algebra

$$fg = (f \wedge g)(f \vee g). \tag{2}$$

Indeed, using the formula $f + g = f \vee g + f \wedge g$ we obtain

$$\begin{aligned} fg - (f \wedge g)(f \vee g) &= fg - (f \wedge g)(f + g - f \wedge g) \\ &= (f - f \wedge g)(g - f \wedge g) = 0. \end{aligned}$$

Using (2) we get $u = (u \wedge e)(u \vee e)$, so we can take $v = u \wedge e$ and $w = u \vee e$.

In [2, Corollary 2.3], the next theorem has been proved by means of representation.

THEOREM 3.7. *The Archimedean f-algebra A with unit element e is uniformly complete iff A is e-uniformly complete.*

PROOF. Obviously, uniform completeness implies e-uniform completeness. For the proof of the converse take a u-uniform Cauchy sequence $\{f_n: n = 1, 2, \ldots\}$ in A. By the preceding proposition $u = vw$, with $0 \leqslant v \leqslant e$ and w^{-1} exists in A. The sequence $\{g_n = w^{-1}f_n: n = 1, 2, \ldots\}$ is a v-uniform, and hence an e-uniform, Cauchy sequence in A. By hypothesis there exists $g \in A$ such that $g_n \to g(e)$. Then $f_n \to wg(w)$ and since $\{f_n: n = 1, 2, \ldots\}$ is a u-uniform Cauchy sequence we conclude that $f_n \to wg(u)$.

We now indicate an elementary construction of the square root. From the next lemma it follows that in a semiprime Archimedean f-algebra a square root (if it exists) is necessarily unique.

LEMMA 3.8. *Let A be a semiprime Archimedean f-algebra. If $0 \leqslant u, v \in A$ then $u \leqslant v$ iff $u^2 \leqslant v^2$.*

PROOF. By (2), proof of Proposition 3.6, we have $uv = (u \wedge v)(u \vee v) = \{u(u \vee v)\} \wedge \{v(u \vee v)\} = (u^2 \vee uv) \wedge (uv \vee v^2) = (u^2 \wedge v^2) \vee (uv)$, so $u^2 \wedge v^2 \leqslant uv$ for $0 \leqslant u, v \in A$. Now suppose that $u^2 \leqslant v^2$ and that $u \leqslant v$ does not hold, i.e., $u \wedge v < u$. This implies $(u \wedge v)^2 < u^2$ (indeed, $u = u \wedge v + w$ for some $w > 0$, so $u^2 \geqslant (u \wedge v)^2 + w^2$ and $w^2 > 0$ since A is semiprime). Hence $u^2 = u^2 \wedge v^2 = u^2 \wedge v^2 \wedge (uv) = (u \wedge v)^2 < u^2$, a contradiction. Hence $u^2 \leqslant v^2$ implies $u \leqslant v$. The converse is obvious.

THEOREM 3.9. *Let A be a uniformly complete f-algebra with unit element e. For any $0 \leqslant u \in A$ there exists a uniquely determined element $0 \leqslant v \in A$ such that $v^2 = u$ ($v = \sqrt{u} = u^{1/2}$).*

PROOF. (1) Suppose $\rho e \leqslant u \leqslant e$ for some $0 < \rho < 1$. Put $u_0 = e$ and $u_{n+1} = u_n + \frac{1}{2}(u - u_n^2)$, $n = 0, 1, \ldots$. Then $\{u_n: n = 0, 1, \ldots\}$ is an e-uniform Cauchy sequence, so $u_n \to v(e)$ for some $0 \leqslant v \in A$. Obviously, $v^2 = u$. The construction of the square root in this case is similar to the construction of the square root of a positive Hermitian operator (see e.g. **[20**, §54]).

(2) Suppose $0 \leqslant u \leqslant e$. Put $u_n = u + n^{-1}e$ $(n = 1, 2, \ldots)$. Since $n^{-1}e \leqslant u_n \leqslant (1 + n^{-1})e$, it follows easily from (1) that $\sqrt{u_n}$ exists $(n = 1, 2, \ldots)$. Moreover, $\{\sqrt{u_n} : n = 1, 2, \ldots\}$ is an e-uniform Cauchy sequence; hence $\sqrt{u_n} \to v(e)$ for some $0 \leqslant v \in A$. Again $v^2 = u$.

(3) Finally, $0 \leqslant u \in A$ arbitrary. Since, by (2), $\sqrt{u \wedge ne}$ exists $(n = 1, 2, \ldots)$ and $u \wedge ne \uparrow u(u^2)$, it is immediate that $\sqrt{u}$ exists in A.

In the next example it is demonstrated that both the condition of uniform completeness and the condition of the unit element are essential in the last theorem.

EXAMPLE 3.10. (i) If A is the f-algebra of Example 3.5, then the element i has no square root in A.

(ii) Let A be the uniformly complete f-algebra consisting of all $f \in C([0, 1])$ for which there exists $n \in \mathbf{N}$ such that $|f| \leqslant ni$. Then A has no unit element. The element $i \in A$ has no square root in A.

The next theorem shows that Henriksen's "property (*)" (see **[12**, §3]) holds in any uniformly complete f-algebra with unit element.

THEOREM 3.11. *Let A be a uniformly complete f-algebra with unit element e. If $0 \leqslant u \leqslant v^2$ for some $0 \leqslant v \in A$, then there exists a (unique) $0 \leqslant w \in A$ such that $0 \leqslant w \leqslant v$ and $u = wv$ (we adhere to calling this property (*)).*

PROOF. For $n = 1, 2, \ldots$, put $w_n = u(v + n^{-1}e)^{-1}$. It is easily verified that $\{w_n: n = 1, 2, \ldots\}$ is an e-uniform Cauchy sequence; hence there exists $0 \leqslant w \in A$ such that $w_n \to w(e)$. From $v + n^{-1}e \to v(e)$ it follows that $w_n(v + n^{-1}e) \to wv$ (r.u.). But $w_n(v + n^{-1}e) = u$ for all n, so $u = wv$. Now $w_n \leqslant v$ for all n implies $w \leqslant v$. Moreover, w is unique. Indeed, suppose that also $u = w'v$, $0 \leqslant w' \leqslant v$. Since A is semiprime, $(w - w')v = 0$ implies $w - w' \perp v$. On the other hand $w - w' \in I_v$, and therefore $w = w'$.

COROLLARY 3.12. *Let A be a uniformly complete f-algebra with unit element e.*

(i) *If $0 \leqslant u \leqslant v$, then there exists a unique $0 \leqslant w \leqslant v$ such that $u^2 = wv$.*

(ii) *If $0 \leqslant u \leqslant v^{1+2^{-n}}$ for some $0 \leqslant v \in A$ and some $n \in \mathbf{N}$, then there exists $0 \leqslant w \in A$ such that $u = wv$ (cf. for the $C(X)$ case* **[10**, 1D]).

(iii) *If $0 \leqslant u \leqslant f^2$ for some $f \in A$, then there exists a unique $t \in A$ whch satisfies $|t| \leqslant |f|$ and $u = tf$.*

PROOF. (i) Obvious.

(ii) Apply the preceding theorem $(n + 1)$ times.

(iii) Since $f^2 = (f^+)^2 + (f^-)^2$, the Riesz decomposition property for Riesz spaces implies that $u = p + q$, $0 \leqslant p \leqslant (f^+)^2$ and $0 \leqslant q \leqslant (f^-)^2$. Hence, $p = vf^+$ for some $0 \leqslant v \leqslant f^+$, and $q = wf^-$ for some $0 \leqslant w \leqslant f^-$. Note that $vf^- = wf^+ = 0$. Putting $t = v - w$, we have $|t| \leqslant |f|$ and $tf = u$.

EXAMPLE 3.13. (i) The condition that A has a unit element cannot be dropped in Theorem 3.11. Take

$$A = \{ri: r \in C([0, 1])\},$$

with $i(x) = x$ for all $x \in [0, 1]$. A simple argument shows that A is a uniformly complete semiprime f-algebra. Take $u(x) = x^2|\sin(1/x)|$, $0 < x \leqslant 1$, $u(0) = 0$, and $v = i$. Then $0 \leqslant u \leqslant v^2$, but u cannot be written as $u = pq$, with $0 \leqslant p, q \in A$. In particular, there does not exist $0 \leqslant w \in A$ such that $u = wv$. Note that in this f-algebra $0 \leqslant u \leqslant v$ still implies the existence of an element $0 \leqslant w \in A$ such that $u^2 = wv$.

(ii) The condition that A is uniformly complete cannot be dropped in Theorem 3.11. Let A be the f-algebra as described in Example 3.5. If $u(x) = 1$ and $v(x) = x + 1$, then $0 \leqslant u \leqslant v^2$, but there is no $0 \leqslant w \in A$ such that $u = wv$.

As observed in the above example, property (*) does not hold in general in a uniformly complete semiprime f-algebra. The next proposition provides in this class of f-algebras a necessary and sufficient condition for property (*) to hold.

PROPOSITION 3.14. *Consider in a semiprime Archimedean f-algebra A the following conditions.*

(i) *Property* (*).

(ii) $\hat{A} = \{\pi_f: f \in A\}$ *is an l-ideal in* Orth(A).

Then (i) $\Rightarrow$ (ii).

Moreover, if A is in addition uniformly complete, then (i) $\Leftrightarrow$ (ii).

PROOF. (i) $\Rightarrow$ (ii). Since $\hat{A}$ is an r-ideal and a Riesz subspace of Orth(A), it suffices to prove that $0 \leqslant \pi \leqslant \pi_v$, $v \in A$ and $\pi \in$ Orth(A), implies that $\pi \in \hat{A}$. It follows from $0 \leqslant \pi v \leqslant \pi_v v = v^2$ that $\pi v = wv$ for some $w \in A$, $0 \leqslant w \leqslant v$, i.e., $\pi_{\pi v} = \pi\pi_v = \pi_w \pi_v$ and $0 \leqslant \pi_w \leqslant \pi_v$. Since π_w is unique in Orth(A) with respect to these properties we obtain $\pi = \pi_w \in \hat{A}$.

Suppose now that A is uniformly complete.

(ii) $\Rightarrow$ (i). If $0 \leqslant u \leqslant v^2$ in A, then $0 \leqslant \pi_u \leqslant \pi_v^2$ in Orth(A), and so, by Theorem 3.11, there exists $0 \leqslant \pi \in$ Orth(A) such that $0 \leqslant \pi \leqslant \pi_v$ and $\pi_u = \pi\pi_v$. Since $\hat{A}$ is an l-ideal in Orth(A) we have $\pi \in \hat{A}$; hence there exists $0 \leqslant w \in A$ such that $\pi = \pi_w$ and so $u = wv$.

Note that the equivalence in Proposition 3.14 remains no longer true if A is not uniformly complete. By way of example, the f-algebra A of Example 3.5 satisfies $\hat{A}$ = Orth(A), since A has a unit element, but does not have property (*). Furthermore, uniform completeness and property (*) are independent properties. Indeed, the f-algebra in Example 3.13(i) is uniformly complete but does not have property (*). Conversely, the f-algebra A consisting of all $f \in C(\mathbf{R}^+)$ for which there exists $x_f \in \mathbf{R}^+$ and $r_f \in \mathbf{R}(X)$ (the quotient field of $\mathbf{R}[X]$) such that $f(x) = r_f(x)$ for all $x \geqslant x_f$, has property (*) but is not uniformly complete.

Next we will present an example of a uniformly complete semiprime f-algebra A satisfying the conditions of Proposition 3.14, but not having a unit element.

EXAMPLE 3.15. Let A be the uniformly complete semiprime f-algebra as defined in Example 3.10(ii). We assert that Orth(A) is f-algebra isomorphic to $C_b((0, 1])$. Indeed, for any $p \in C_b((0, 1])$ and $f \in A$ we define the function $\varphi(p)f$ by $\varphi(p)f(x) = p(x)f(x)$ if $x \in (0, 1]$ and $\varphi(p)f(0) = 0$. Then $\varphi(p)f \in A$, and the mapping $\varphi(p)$ from A into itself is an orthomorphism. Hence φ is a mapping from $C_b((0, 1])$ into Orth(A), which clearly is an injective f-algebra homomorphism. We will show that it is actually an isomorphism. Take any $\pi \in$ Orth(A) and define $p(x) = (\pi i)(x)x^{-1}$, $x \in (0, 1]$. Then $p \in C_b((0, 1])$. Since $\varphi(p)i = \pi i$ and since the kernel of an orthomorphism is a band, the fact that i is a weak unit in A implies that $\varphi(p) = \pi$. Hence, φ is onto. Under this isomorphism $\hat{A}$ corresponds to the l-ideal $\{p \in C_b((0, 1]): |p| \leqslant ni$ for some $n \in \mathbf{N}\}$ in $C_b((0, 1])$. Therefore $\hat{A}$ is an l-ideal in Orth(A).

Let $I = \{ri: r \in C([0, 1])\}$; then I is an r-ideal in A, which is uniformly closed in A, but I is not an l-ideal. This is in contrast to the fact that any uniformly closed o-ideal in an Archimedean f-algebra is an l-ideal (see Proposition 3.1(i)). However, it can be proved that every uniformly closed r-ideal in an Archimedean f-algebra with unit element is an l-ideal. Using Theorem 3.11 it is easily checked that $I_{i^2} \subset I$, and so the uniform closure $I_{i^2}^-$ of I_{i^2} in A satisfies $I_{i^2}^- \neq I_i^- = A$ (cf. Proposition 3.2(ii), where it is proved that $I_{f^2}^- = I_f^-$ for all f in a unitary Archimedean f-algebra).

Finally we will prove in this section a multiplicative analogue for f-algebras of the Riesz decomposition property in Riesz spaces. We are indebted to N. G. de Bruijn for his suggestion in the proof.

THEOREM 3.16. *In a uniformly complete semiprime f-algebra A with property* $(*)$ (*in particular if A is uniformly complete with unit element*), $0 \leqslant u \leqslant vw$ *with* $0 \leqslant v$, $w \in A$, *implies that* $u = pq$, *for some* $0 \leqslant p \leqslant v$ *and* $0 \leqslant q \leqslant w$.

PROOF. First assume that A has, in addition, a unit element e. Then, by Theorem 3.9, $z^{1/2}$ exists for any $0 \leqslant z \in A$. Then

$$p = \tfrac{1}{2}(v - w) + \left\{u + \tfrac{1}{4}(v - w)^2\right\}^{1/2} \quad \text{and} \quad q = \tfrac{1}{2}(w - v) + \left\{u + \tfrac{1}{4}(v - w)^2\right\}^{1/2}$$

satisfy $u = pq$ and $0 \leqslant p \leqslant v$, $0 \leqslant q \leqslant w$.

We proceed now to the general case. Since Orth(A) is a uniformly complete f-algebra with unit element, the above observations applied to Orth(A) yield $\pi_u = \pi_1\pi_2$ for some $\pi_1, \pi_2 \in$ Orth(A) with $0 \leqslant \pi_1 \leqslant \pi_v$ and $0 \leqslant \pi_2 \leqslant \pi_w$. By hypothesis, $\hat{A}$ is an l-ideal in Orth(A); therefore $\pi_1, \pi_2 \in \hat{A}$, i.e. $\pi_1 = \pi_p$ and $\pi_2 = \pi_q$ for some $p, q \in A$ with $0 \leqslant p \leqslant v$ and $0 \leqslant q \leqslant w$. Then $u = pq$.

Theorem 3.16 no longer holds if property $(*)$ is omitted. This is illustrated by Example 3.13(i).

4. Ideal theory. We start with some definitions and elementary properties of ideals in f-algebras (see [**12**, **17**]). If I and J are r-ideals, then we denote by IJ the product of I and J, i.e.,

$$IJ = \left\{ \sum_{k=1}^{n} r_k s_k : r_k \in I, s_k \in J, n \in \mathbf{N} \right\}.$$

DEFINITION 4.1. Let I be an r-ideal in the f-algebra A.

(i) I is called semiprime whenever $I = \sqrt{I} = \{f \in A : f^n \in I \text{ for some } n \in \mathbf{N}\}$.

(ii) I is called idempotent whenever $I = I^2$.

(iii) I is called pseudoprime whenever $fg = 0$ implies $f \in I$ or $g \in I$.

(iv) I is called square-root closed whenever for any $f \in I$ there exists $g \in I$ such that $|f| = g^2$.

The following proposition has also been proved by Subramanian [**24**].

PROPOSITION 4.2. *Let A be an Archimedean f-algebra.*

(i) *If I is an l-ideal in A, then $\sqrt{I}$ is an l-ideal.*

(ii) *If P is an l-ideal in A then P is a prime r-ideal iff P is pseudoprime and semiprime.*

PROOF. (i) As is well known, $\sqrt{I}$ is an r-ideal. Moreover,

$$f \in \sqrt{I} \Leftrightarrow f^2 = |f|^2 \in \sqrt{I} \Leftrightarrow |f| \in \sqrt{I} .$$

Finally, if $0 \leqslant u \leqslant v$ in A with $v^k \in I$, then $u^k \in I$, so $u \in \sqrt{I}$. This implies that $\sqrt{I}$ is an l-ideal.

(ii) A prime r-ideal is obviously pseudoprime and semiprime. Conversely, let P be a pseudoprime and semiprime l-ideal and suppose $fg \in P$. It follows from $(|f| \wedge |g|)^2 \leqslant |f| \cdot |g| = |fg| \in P$ that $|f| \wedge |g| \in \sqrt{P} = P$. Since

$$(|f| - |f| \wedge |g|) \cdot (|g| - |f| \wedge |g|) = 0,$$

either $|f| - |f| \wedge |g| \in P$ or $|g| - |f| \wedge |g| \in P$. Combining this with $|f| \wedge |g| \in P$, we find $|f| \in P$ or $|g| \in P$, and so $f \in P$ or $g \in P$.

PROPOSITION 4.3. *Let A be an Archimedean semiprime f-algebra.*

(i) *If I is a square-root closed r-ideal in A, then $I = I^2$ and $I = \sqrt{I}$.*

(ii) *Every idempotent l-ideal in A is semiprime.*

PROOF. (i) We first prove $I = I^2$. For any $f \in I$ there exists $g \in I$ such that $|f| = g^2$, so $|f| \in I$. Hence $f^+, f^- \in I$, which implies $f^+ = p^2$, $f^- = q^2$ for some $p, q \in I$. Therefore $f = (p + q)(p - q) \in I^2$. Secondly we show that $I = \sqrt{I}$. It is sufficient to prove that $f \in A, f^2 \in I$ implies that $f \in I$. By hypothesis, $f^2 = g^2$ for some $g \in I$. Using Lemma 3.8, we deduce $|f| = |g|$. As observed above, $g \in I$ implies $|g| \in I$, so $|f| \in I$. Now it follows from $(f^+)^2 = f^+|f| \in I$ that $(f^+)^2 = h^2$ for some $h \in I$, and therefore $f^+ = |h| \in I$. We conclude, finally, that $f = 2f^+ - |f| \in I$.

(ii) Suppose that I is an l-ideal for which $I = I^2$. It suffices to prove that $0 \leqslant u \in A$, $u^2 \in I$ implies $u \in I$. Since $u^2 \in I = I^2$, we have $u^2 = \sum_{k=1}^n r_k s_k$ for some $r_k, s_k \in I$, $n \in \mathbf{N}$. Then $u^2 \leqslant \sum_{k=1}^n |r_k|\,|s_k| \leqslant pq$, with $p = \sum_{k=1}^n |r_k| \in I$ and $q = \sum_{k=1}^n |s_k| \in I$. Hence $u^2 \leqslant (p + q)^2$, which implies $u \leqslant p + q$, so $u \in I$.

EXAMPLE 4.4. We present an example of an r-ideal I in an Archimedean semiprime f-algebra, for which $I = I^2$ and $I = \sqrt{I}$, but I is not square-root closed. Let R_1 be the commutative unitary algebra consisting of all real functions

$$p(x) = \sum_{k=1}^{n} a_k x^{r_k}, \qquad a_k \in \mathbf{R}, 0 \leqslant r_k \in \mathbf{Q}, n \in \mathbf{N},$$

and let A be the set of all $f \in C(\mathbf{R}^+)$ for which there exists $x_f \in \mathbf{R}^+$ and $p_f \in R_1$ such that $f(x) = p_f(x)$ for all $x \geqslant x_f$. Then A is an Archimedean f-algebra with unit element, but A is not uniformly complete. Take $I = \{f \in A: p_f(0) = 0\}$. Then I is a maximal r-ideal in A, so $I = \sqrt{I}$. We assert that $I = I^2$. Indeed, if $f \in I$, then $p_f \in I^2$. Hence it suffices to prove that $g = f - p_f \in I^2$. From $g^+(x) = g^-(x) = 0$ for all $x \geqslant x_f$, it follows that $\sqrt{g^+}$, $\sqrt{g^-} \in I$, and therefore

$$g = (\sqrt{g^+})^2 - (\sqrt{g^-})^2 \in I^2.$$

If the function w is defined by $w(x) = x^2 + x$, then $w \in I$, but there does not even exist an element $0 \leqslant u \in A$ such that $u^2 = w$ (observe that the function $\sqrt{x^2 + x}$ is not a member of R_1). Therefore I is not square-root closed. It is worthwhile to observe that I is not an l-ideal in A. Indeed, if we take the function u defined by $u(x) = x$, $0 \leqslant x \leqslant 1$, and $u(x) = 1$, $x \geqslant 1$, then $u(x) \leqslant i(x) = x$ for all $x \in \mathbf{R}^+$, $i \in I$, but $u \notin I$.

EXAMPLE 4.5. An example of an r-ideal in I in an Archimedean semiprime f-algebra, for which $I = I^2$, but $I \neq \sqrt{I}$. We take R_1 as in the preceding example, and denote by v the function $v(x) = \sqrt{x^2 + x}$. The commutative unitary algebra R_2 is defined by

$$R_2 = \{p: p = s + tv \text{ with } s, t \in R_1\}$$

(note that s and t are uniquely determined by p). Let A be the Archimedean unitary f-algebra of all $f \in C(\mathbf{R}^+)$ for which there exist $x_f \in \mathbf{R}^+$ and $p_f \in R_2$ such that $f(x) = p_f(x)$ for all $x \geqslant x_f$. Take $I = \{f \in A: p_f = s + tv$ with $s, t \in R_1$ and $s(0) = t(0) = 0\}$. Since $v^2 \in I$ and $v \notin I$ we have $I \neq \sqrt{I}$. The proof that $I^2 = I$ is similar to Example 4.4.

EXAMPLE 4.6. An example of an l-ideal I in a uniformly complete semiprime f-algebra, for which $I = \sqrt{I}$, but $I \neq I^2$. Let A be the f-algebra as in Examples 3.10(ii) and 3.15. Then $I = \{f \in A: f(\frac{1}{2}) = 0\}$ is an l-ideal as well as a prime r-ideal, and hence I is a semiprime r-ideal. Since $I^2 \subset \{f \in I_{i^2}: f(\frac{1}{2}) = 0\}$ it is obvious that $I^2 \neq I$.

In general, the radical $\sqrt{I}$ of an r-ideal I is not an l-ideal (see Example 4.4). However, the situation turns out to be nicer in the uniformly complete case.

THEOREM 4.7. *If I is an r-ideal in the uniformly complete semiprime f-algebra A, then $\sqrt{I}$ is an l-ideal* (*equivalently, every semiprime r-ideal is an l-ideal*).

PROOF. As observed before, $f \in \sqrt{I}$ iff $|f| \in \sqrt{I}$. It remains to prove that $0 \leqslant u \leqslant v$, $v \in \sqrt{I}$, implies $u \in \sqrt{I}$. Since $\mathrm{Orth}(A)$ is a uniformly complete f-algebra with unit element, it follows from $0 \leqslant \pi_u \leqslant \pi_v$ and from Corollary 3.12(i) that $\pi_u^2 = \pi\pi_v$ for some $\pi \in \mathrm{Orth}(A)$, $0 \leqslant \pi \leqslant \pi_v$. Applying this property once more, we find $\pi^2 = \pi_1\pi_v$, $0 \leqslant \pi_1 \in \mathrm{Orth}(A)$. Hence, $\pi_u^4 = (\pi_1\pi_v)\pi_v^2 = \pi_{\pi_1 v} \cdot \pi_v^2$, and so $u^4 = (\pi_1 v)v^2 \in \sqrt{I}$, and therefore $u \in \sqrt{I}$.

COROLLARY 4.8 (CF. [13, LEMMA 1.5]). *If P is a prime r-ideal in the uniformly complete semiprime f-algebra A, then P is an l-ideal* (*and hence P is a prime o-ideal*).

Observe that this corollary does not necessarily hold in a nonuniformly complete f-algebra (see again Example 4.4).

If I is an r-ideal in the f-algebra A, then I^2 is in general not an l-ideal. By way of example, take $A = C([0, 1])$ and $I = (i)$. We have, however, the following theorem.

THEOREM 4.9. *In a uniformly complete semiprime f-algebra A*

(i) *every idempotent r-ideal is an l-ideal, and*

(ii) *every idempotent r-ideal is semiprime.*

PROOF. (i) Suppose that I is an r-ideal in A for which $I = I^2$. Put $\hat{I} = \{\pi_f: f \in I\}$. Then $\hat{I}$ is an r-ideal in $\hat{A}$. We first show that $\hat{I}$ is an r-ideal in $\mathrm{Orth}(A)$. To this end, take $\pi_f \in \hat{I}$, $\pi \in \mathrm{Orth}(A)$. By hypothesis $f = \sum_{k=1}^n r_k s_k$, $r_k, s_k \in I$ $(k = 1, \ldots, n)$, and so

$$\pi\pi_f = \pi\left(\sum_{k=1}^n \pi_{r_k}\pi_{s_k}\right) = \sum_{k=1}^n (\pi\pi_{r_k})\pi_{s_k} = \sum_{k=1}^n \pi_{\pi r_k} \cdot \pi_{s_k} \in \hat{I},$$

since $\hat{I}$ is an r-ideal in $\hat{A}$.

If we prove that $\hat{I}$ is an l-ideal in $\mathrm{Orth}(A)$, then we are done. For this purpose it suffices to prove that $0 \leqslant \pi \leqslant |\pi_f|$, $\pi \in \mathrm{Orth}(A)$ and $f \in I$, implies $\pi \in \hat{I}$. Since $I = I^2$, $|f| \leqslant \sum_{k=1}^n |r_k s_k|$, $r_k, s_k \in I$ $(k = 1, \ldots, n)$; hence $0 \leqslant \pi \leqslant |\pi_f| = \pi_{|f|} \leqslant \sum_{k=1}^n |\pi_{r_k}\pi_{s_k}| \leqslant \sum_{k=1}^n (\pi_{r_k}^2 + \pi_{s_k}^2)$. By the Riesz decomposition property, $\pi = \sum_{k=1}^n (\rho_k + \sigma_k)$ for some $\rho_k, \sigma_k \in \mathrm{Orth}(A)$ with $0 \leqslant \rho_k \leqslant \pi_{r_k}^2$ and $0 \leqslant \sigma_k \leqslant \pi_{s_k}^2$ $(k = 1, \ldots, n)$. Using Corollary 3.9(iii) and the fact that $\hat{I}$ is an r-ideal in $\mathrm{Orth}(A)$, we conclude that $\rho_k, \sigma_k \in \hat{I}$ $(k = 1, \ldots, n)$, and so $\pi \in \hat{I}$.

(ii) Straightforward from (i) and Proposition 4.3(ii).

THEOREM 4.10. *For an r-ideal I in a uniformly complete semiprime f-algebra A the following conditions are equivalent.*

(i) *I is square-root closed.*

(ii) $I = I^2$.

PROOF. (i) $\Rightarrow$ (ii). See Proposition 4.3(i).

(ii) $\Rightarrow$ (i). As observed in the proof of the preceding theorem $I = I^2$ in A implies that $\hat{I}$ is an l-ideal in $\mathrm{Orth}(A)$. By (ii) of the same theorem, $\hat{I} = \sqrt{\hat{I}}$ in $\mathrm{Orth}(A)$. Hence, $f \in I$ implies that $\sqrt{\pi_{|f|}} \in \hat{I}$, i.e., there exists $0 \leqslant u \in I$ such that $\sqrt{\pi_{|f|}} = \pi_u$. In other words, there exists $0 \leqslant u \in I$ such that $|f| = u^2$.

We emphasize the fact that both conditions of the last theorem are, in general, stronger than the condition that $I = \sqrt{I}$ (see Example 4.6). It seems interesting therefore to state a necessary and sufficient condition for the converse to hold.

THEOREM 4.11. *In a uniformly complete semiprime f-algebra A the following statements are equivalent*:

(i) *Every semiprime r-ideal is idempotent.*

(ii) *For every* $0 \leqslant u \in A$ *there exists* $0 \leqslant v \in A$ *such that* $u = v^2$ (*i.e. A is square-root closed*).

PROOF. (i) $\Rightarrow$ (ii). From the hypothesis it follows in particular that $A = A^2$, and so, by Theorem 4.10, A is square-root closed.

(ii) $\Rightarrow$ (i). Let I be a semiprime r-ideal in A. For any $0 \leqslant u \in I$ there exists $0 \leqslant v \in A$ such that $u = v^2$. Since $I = \sqrt{I}$ we deduce that $v \in I$, hence $u \in I^2$.

It is worthwhile to note that if A is a square-root closed uniformly complete semiprime f-algebra, then $\hat{A}$ is an idempotent r-ideal in Orth(A) and so, by Theorem 4.9(i), $\hat{A}$ is an l-ideal in Orth(A). It is shown in Examples 3.10(ii) and 3.15 that the converse does not hold.

Combining Theorems 4.9, 4.10 and 4.11 we get the following theorem.

THEOREM 4.12 (CF. FOR THE $C(X)$ CASE [**25**, THEOREM 2.1]; FOR THE GENERAL CASE COMPARE [**12**, THEOREMS 3.4 AND 3.14]). *For an r-ideal I in a uniformly complete semiprime square-root closed f-algebra A the following conditions are equivalent.*

(i) *I is semiprime.*

(ii) *I is idempotent.*

(iii) *I is square-root closed.*

In particular these three conditions are equivalent in a uniformly complete f-algebra with unit element.

In [**25**, Corollary 2.13], it is proved that I^2 is an l-ideal for any l-ideal I in $C(X)$. In the next theorem we generalize this result.

THEOREM 4.13. *If I and J are l-ideals in the Archimedean f-algebra A with property* ($*$), *then IJ is again an l-ideal. In fact*

$$IJ = \{ f \in A \colon |f| \leqslant uv \text{ for some } 0 \leqslant u \in I \text{ and } 0 \leqslant v \in J \}.$$

PROOF. It is sufficient to prove that $0 \leqslant u \leqslant |f|, f \in IJ$, implies $u \in IJ$. From $f \in IJ$ it follows that $|f| \leqslant pq$ with $0 \leqslant p \in I$ and $0 \leqslant q \in J$. Hence $0 \leqslant u \leqslant (p + q)^2$. Property ($*$) implies that $u = w(p + q)$ for some $0 \leqslant w \in A$. If we put $w_1 = w \wedge q$ and $w_2 = w \wedge p$, then $w_1 \in J$, $w_2 \in I$ and $w_1 p + w_2 q = wp \wedge pq + wq \wedge pq = wp + wq = u$, so $u \in IJ$.

We show in the next example that property ($*$) in Theorem 4.13 is not superfluous.

EXAMPLE 4.14. Take $A = \{ri \colon r \in C([0, 1])\}$ as in Example 3.13(i). Consider the l-ideal $I = \{f \in A \colon f(\frac{1}{2}) = 0\}$. It is an easy matter to verify that $I^2 = \{ri^2 \colon r \in C([0, 1]), r(\frac{1}{2}) = 0\}$ is not an l-ideal.

D. Rudd proves in [**22**] that the sum of two semiprime r-ideals in $C(X)$ is again semiprime. The following theorem generalizes this result.

THEOREM 4.15. *Let A be a uniformly complete semiprime f-algebra. If $\hat{A}$ is an l-ideal in* Orth(A) (*in particular if A has a unit element*), *then the sum of any two semiprime r-ideals is semiprime.*

PROOF. First assume that A has a unit element. If I and J are semiprime r-ideals in A, then, by Theorem 4.12, $I = I^2$ and $J = J^2$. By the same theorem it is sufficient to prove that $(I + J)^2 = I + J$. Let $0 \leqslant u \in I + J$. Then $u = v + w$, $0 \leqslant v \in I^2$ and $0 \leqslant w \in J^2$. Since I and J are l-ideals, we have $0 \leqslant v \leqslant pq$ $(0 \leqslant p, q \in I)$ and $0 \leqslant w \leqslant rs$ $(0 \leqslant r, s \in J)$. Therefore $u \leqslant pq + rs \leqslant (p + r) \cdot (q + s) \leqslant (p + q + r + s)^2$. By Theorem 3.11, there exists $z \in A$ such that $0 \leqslant z \leqslant p + q + r + s$ and $u = z(p + q + r + s)$. This implies $u \in (I + J)^2$, and since $I + J$ is an l-ideal it follows that $I + J \subset (I + J)^2$. Hence, the theorem holds in the case that A has a unit element.

Secondly assume that $\hat{A}$ is an l-ideal in Orth(A). For any r-ideal I in A we denote $\hat{I} = \{\pi_f : f \in I\}$, which is an r-ideal in $\hat{A}$. If I is semiprime, then $\hat{I}$ is semiprime in $\hat{A}$ and moreover $\hat{I}$ is an r-ideal in Orth(A). Indeed, if $\pi_f \in \hat{I}$ and $\pi \in \text{Orth}(A)$, then $\pi^2\pi_f = \pi_{\pi^2 f} \in \hat{A}$, so $(\pi\pi_f)^2 = \pi_{\pi^2 f}\pi_f \in \hat{I}$, and since $\pi\pi_f \in \hat{A}$ and $\hat{I}$ is semiprime, we have $\pi\pi_f \in \hat{I}$. Now let I and J be semiprime r-ideals in A. The first part of the present proof, applied to Orth(A), shows that $\overset{\bullet}{\sqrt{\hat{I} + \hat{J}}} = \overset{\bullet}{\sqrt{\hat{I}}} + \overset{\bullet}{\sqrt{\hat{J}}}$, where $\overset{\bullet}{\sqrt{\hat{I}}}$ denotes the radical of $\hat{I}$ in Orth(A) (whereas $\sqrt{\hat{I}}$ denotes the radical of $\hat{I}$ in $\hat{A}$). Using that the lattice of all l-ideals in Orth(A) is distributive, we derive that

$$\begin{aligned}\sqrt{\hat{I} + \hat{J}} &= \left(\overset{\bullet}{\sqrt{\hat{I} + \hat{J}}}\right) \cap \hat{A} = \left(\overset{\bullet}{\sqrt{\hat{I}}} + \overset{\bullet}{\sqrt{\hat{J}}}\right) \cap \hat{A} \\ &= \overset{\bullet}{\sqrt{\hat{I}}} \cap \hat{A} + \overset{\bullet}{\sqrt{\hat{J}}} \cap \hat{A} = \sqrt{\hat{I}} + \sqrt{\hat{J}} = \hat{I} + \hat{J}.\end{aligned}$$

Hence $\hat{I} + \hat{J}$ is semiprime in $\hat{A}$, and so $I + J$ is semiprime in A.

In the next example we will show that neither the condition of uniform completeness nor the condition that $\hat{A}$ is an l-ideal in Orth(A) can be dropped in the preceding theorem.

EXAMPLE 4.16. (i) In $C([0, 1])$ we denote by i the function $i(x) = x$ and $w = \sqrt{i}$, and by I_i we denote the o-ideal generated in $C([0, 1])$ by i. Let $A = \{f \in C([0, 1]): f = \alpha e + \beta w + g,\ g \in I_i \text{ and } \alpha, \beta \in \mathbf{R}\}$. Then A is a nonuniformly complete Archimedean f-algebra with unit element e. Let

$$\begin{aligned} I &= \left\{f \in A : f\left((2n - 1)^{-1}\right) = 0 \text{ for } n = 1, 2, \ldots\right\}, \\ J &= \left\{f \in A : f\left((2n)^{-1}\right) = 0 \text{ for } n = 1, 2, \ldots\right\}. \end{aligned}$$

Evidently I and J are semiprime l-ideals in A. We claim, however, that $I + J$ is not semiprime. Clearly $w^2 = i \in I + J$, but we will show that $w \notin I + J$. Indeed, suppose on the contrary that $w \in I + J$. Then $w = u + v$ for some $0 \leqslant u \in I$ and $0 \leqslant v \in J$. Since $u(0) = 0$ the function u can be written as $u = \beta w + g$, $\beta \in \mathbf{R}$, $g \in I_i$. If $\beta \neq 0$, there would exist an interval $(0, \varepsilon]$ such that $u(x) \neq 0$ for all $x \in (0, \varepsilon]$, which is at variance with $u \in I$. Hence $\beta = 0$, and so $u = g \in I_i$. Analogously $v \in I_i$. Combining these results we would have $w \in I_i$, which is contradictory as well, so $w \notin I + J$.

(ii) Let A be the f-algebra as introduced in Example 3.13(i). As observed there, A is uniformly complete and semiprime. It is routine to prove that Orth(A) is f-algebra isomorphic to $C([0, 1])$, and under this isomorphism $\hat{A}$ corresponds to A.

Hence $\hat{A}$ is not an l-ideal in Orth(A). Let the saw-tooth functions $0 \leqslant u, v \in C([0, 1])$ be defined by

$$u(x) = \begin{cases} 0 & \text{for } x = 1/n, n = 1, 3, 5, \ldots, \\ 1/n & \text{for } x = 1/n, n = 2, 4, 6, \ldots, \\ \text{linear} & \text{in between,} \end{cases}$$

$$v(x) = \begin{cases} 1/n & \text{for } x = 1/n, n = 1, 3, 5, \ldots, \\ 0 & \text{for } x = 1/n, n = 2, 4, 6, \ldots, \\ \text{linear} & \text{in between,} \end{cases}$$

so $u + v = i$. Putting $I = \{f \in A : f(1/n) = 0, n = 1, 3, 5, \ldots\}$ and $J = \{f \in A : f(1/n) = 0, n = 2, 4, 6, \ldots\}$, I and J are semiprime l-ideals in A. Furthermore, $ui \in I$, $vi \in J$ and $ui + vi = i^2$ imply that $i \in \sqrt{I + J}$. It is not difficult to show that $i \notin I + J$, so $I + J$ is not semiprime.

Theorem 4.15, combined with Example 4.16, gives a partial answer to "question A" posed by M. Henriksen in **[12]**. Note that if A is uniformly complete with unit element, every l-ideal in A is "square dominated" (a notion introduced by Henriksen in the same paper, §3). Hence, the result of the preceding theorem follows in this case also from Theorem 3.9 of that paper.

We now turn our attention to prime r-ideals. The next theorem is proved for $C(X)$ by L. Gillman and C. W. Kohls **[11**, 4.2(a)].

THEOREM 4.17. *If P is an r-ideal in the uniformly complete semiprime f-algebra A, then P is a prime r-ideal iff P is pseudoprime and semiprime.*

PROOF. Suppose that P is a pseudoprime r-ideal for which $P = \sqrt{P}$. By Theorem 4.7, P is an l-ideal and by Proposition 4.2(ii), P is prime. The converse being clear, the theorem is proved.

In the next example we show that Theorem 4.17 remains no longer true if A is not uniformly complete.

EXAMPLE 4.18. Let A be the Archimedean unitary f-algebra of all $f \in C(\mathbf{R}^+)$ for which there exist $x_f \in \mathbf{R}^+$ and $p_f \in \mathbf{R}[X]$ such that $f(x) = p_f(x)$ for all $x \geqslant x_f$. Take $p \in \mathbf{R}[X]$ such that p is the product of at least two different irreducible polynomials. Defining $I = \{f \in A : p_f \in (p)\}$, it follows from $(p) = \sqrt{(p)}$ that $I = \sqrt{I}$. Since $\mathbf{R}[X]$ contains no divisors of zero, it is easily seen that I is pseudoprime. By construction, I is not a prime r-ideal.

The following theorem generalizes a result of R. D. Williams **[25**, Corollary 2.2]. The proof follows directly from Theorems 4.12 and 4.17.

THEOREM 4.19. *Let A be a uniformly complete semiprime f-algebra. If A is square-root closed, then the r-ideal P is prime iff P is pseudoprime and idempotent.*

Example 4.6 illustrates that Theorem 4.19 does not hold in a non-square-root closed f-algebra.

5. The range of an orthomorphism. In general the range R_π of an orthomorphism π on an Archimedean Riesz space L is not an o-ideal. By way of example, take $L = C([0, 1])$ and $\pi f = if$ for all $f \in L$. It is well known that for a positive orthomorphism π the range R_π is an o-ideal whenever L is a Dedekind complete Riesz space (see [**4**, Proposition 1]). In this section we will prove this result for arbitrary orthomorphisms under the much weaker condition that L is uniformly complete and normal. This result will be applied to ideal theory in f-algebras in the next section. As a special case we get that the range of an orthomorphism on a Dedekind σ-complete Riesz space is an o-ideal.

We first need a lemma.

LEMMA 5.1. *Let L be a normal Archimedean Riesz space, $0 \leqslant \pi \in \operatorname{Orth}(L)$, $0 \leqslant v \in L$, and $g \in L$ such that $|g| \leqslant \pi v$. Then there exists an element $f \in L$ which satisfies $|f| \leqslant \frac{1}{3} v$ and $|g - \pi f| \leqslant \frac{2}{3}\pi v$.*

PROOF. From $(g + \frac{1}{3}\pi v)^- \wedge (g - \frac{1}{3}\pi v)^+ = 0$, and from the normality of L it follows that

$$L = \left\{\left(g - \tfrac{1}{3}\pi v\right)^+\right\}^d + \left\{\left(g + \tfrac{1}{3}\pi v\right)^-\right\}^d.$$

Therefore $v = v_1 + v_2$ with $0 \leqslant v_1 \in \{(g - \frac{1}{3}\pi v)^+\}^d$, $0 \leqslant v_2 \in \{(g + \frac{1}{3}\pi v)^-\}^d$. We assert that $f = \frac{1}{3}(v_2 - v_1)$ satisfies the conditions of the lemma. Evidently $|f| \leqslant \frac{1}{3} v$. Furthermore,

$$\begin{aligned}\left(g - \pi f - \tfrac{2}{3}\pi v\right)^+ &= \left(g - \tfrac{1}{3}\pi v - \tfrac{2}{3}\pi v_2\right)^+ = \left(g - \pi v + \tfrac{2}{3}\pi v_1\right)^+ \\ &\quad \wedge \left(g - \tfrac{1}{3}\pi v - \tfrac{2}{3}\pi v_2\right)^+ \\ &\leqslant \left(\tfrac{2}{3}\pi v_1\right) \wedge \left(g - \tfrac{1}{3}\pi v\right)^+ = 0,\end{aligned}$$

since $v_1 \perp (g - \frac{1}{3}\pi v)^+$ and π is an orthomorphism. This implies $g - \pi f \leqslant \frac{2}{3}\pi v$. Similarly it is shown that $g - \pi f \geqslant -\frac{2}{3}\pi v$, and the lemma is proved.

THEOREM 5.2. *In a uniformly complete normal Riesz space the range of every positive orthomorphism is an o-ideal.*

PROOF. Take $0 \leqslant \pi \in \operatorname{Orth}(L)$. Since the range R_π is a Riesz subspace of L it suffices to prove that $0 \leqslant u \leqslant \pi v$ $(0 \leqslant v \in L)$ implies that $u \in R_\pi$. We shall define inductively elements $z_n \in L$ $(n = 0, 1, \ldots)$ with the properties

(i) $|z_{n-1} - z_n| \leqslant (\frac{2}{3})^n v$, $n = 1, 2, \ldots,$

(ii) $|u - \pi z_n| \leqslant (\frac{2}{3})^n \pi v$, $n = 0, 1, \ldots,$

as follows. Put $z_0 = 0$, and suppose that $z_0, \ldots, z_n$ $(n \geqslant 0)$ with properties (i) and (ii) have been defined. Since $|u - \pi z_n| \leqslant \pi((\frac{2}{3})^n v)$, Lemma 5.1 implies that there exists $f_{n+1} \in L$ such that $|f_{n+1}| \leqslant \frac{1}{3}(\frac{2}{3})^n v \leqslant (\frac{2}{3})^{n+1} v$ and such that

$$|u - \pi z_n - \pi f_{n+1}| \leqslant \tfrac{2}{3}\pi\left(\left(\tfrac{2}{3}\right)^n v\right) = \left(\tfrac{2}{3}\right)^{n+1} \pi v.$$

Taking $z_{n+1} = z_n + f_{n+1}$, the induction step is concluded.

Since L is uniformly complete and $\{z_n \colon n = 1, 2, \ldots\}$ is a v-uniform Cauchy sequence, there exists $z \in L$ such that $z_n \to z(v)$, and therefore $\pi z_n \to \pi z(\pi v)$. By property (ii), $\pi z_n \to u(\pi v)$; hence $\pi z = u$, i.e., $u \in R_\pi$.

COROLLARY 5.3. *In a uniformly complete normal Riesz space the range of every orthomorphism is an o-ideal.*

PROOF. Let $\pi \in \mathrm{Orth}(L)$ and suppose that $0 \leqslant g \leqslant |\pi f|$. Since $|\pi f| = |\pi|\,|f|$ and since $R_{|\pi|}$ is an o-ideal by the preceding theorem, we have $0 \leqslant g \in R_{|\pi|}$, i.e., there exists $0 \leqslant h \in L$ such that $g = |\pi|h$. Now it follows from $|\pi|h = |\pi h|$ that $\pi h \in R_{|\pi|}$; hence there exists $q \in L$ such that $\pi h = |\pi|q$. This implies that $\pi^+(h - q) = \pi^-(h + q)$, and since $\pi^+(h - q) \perp \pi^-(h + q)$, we have $\pi^+(h - q) = \pi^-(h + q) = 0$, i.e., $\pi^+h = \pi^+q$ and $\pi^-h = -\pi^-q$. Hence, $g = |\pi|h = \pi^+h + \pi^-h = \pi^+q - \pi^-q = \pi q$, and so $g \in R_\pi$. Now suppose that $|g| \leqslant |\pi f|$. Then it follows from the above that $g^+, g^- \in R_\pi$, hence $g \in R_\pi$. We conclude that R_π is an o-ideal.

REMARK 5.4. (i) The authors have proved in [**16**, Theorem 9.15], that an Archimedean Riesz space L is uniformly complete and normal iff L has the σ-interpolation property (i.e., if $f_n\uparrow \leqslant g_n\downarrow$ in L, then there exists $h \in L$ such that $f_n \leqslant h \leqslant g_n$ for all n). This is a generalization of a result of G. L. Seever [**23**], who has proved this for the case that $L = C(X)$ for some compact Hausdorff space X.

(ii) In Example 6.7 we shall show that the condition of uniform completeness cannot be dropped in Theorem 5.2. The example at the beginning of this section shows that the normality cannot be deleted.

6. Normal f-algebras. The main purpose in this section is to find necessary and sufficient conditions in order that every r-ideal in an Archimedean f-algebra is an l-ideal.

For the $C(X)$ case such conditions are well known. In [**10**, Theorem 14.25] it is proved that in $C(X)$ the following conditions are equivalent.

(α) Every r-ideal in $C(X)$ is an l-ideal.

(β) X is an F-space (i.e., every finitely generated r-ideal in $C(X)$ is principal).

(γ) For any $f \in C(X)$ the sets pos f and neg f are completely separated.

Condition (γ) can be reformulated by saying that $C(X) = \{f^+\}^d + \{f^-\}^d$ for every $f \in C(X)$, i.e., $C(X)$ is a normal Riesz space (see [**16**, Theorem 10.5]). It seems interesting therefore to consider normal f-algebras more closely.

We refer the reader in this connection also to [**24**, §4].

PROPOSITION 6.1. *Let A be an Archimedean semiprime f-algebra. If every r-ideal in A is an l-ideal, then A is normal.*

PROOF. We have to prove that $A = \{f^+\}^d + \{f^-\}^d$ for every $f \in A$. The r-ideal (f) is an o-ideal, so $|f| = rf + \alpha f$ for appropriate $r \in A$, $\alpha \in \mathbf{R}$. Now it follows from $f^+ - rf^+ - \alpha f^+ = -f^- - rf^- - \alpha f^-$ and from $f^+ - rf^+ - \alpha f^+ \perp -f^- - rf^- - \alpha f^-$ that $f^+ - rf^+ - \alpha f^+ = 0$ and $f^- + rf^- + \alpha f^- = 0$.

Let $g \in A$ be arbitrary and define $g_1 = \frac{1}{2}(g - rg - \alpha g)$, $g_2 = \frac{1}{2}(g + rg + \alpha g)$. Since $g_1 f^+ = \frac{1}{2}(g - rg - \alpha g)f^+ = \frac{1}{2}g(f^+ - rf^+ - \alpha f^+) = 0$, we have $g_1 \in \{f^+\}^d$. Analogously, $g_2 \in \{f^-\}^d$, and therefore $g = g_1 + g_2 \in \{f^+\}^d + \{f^-\}^d$.

The next example shows that even in a Dedekind complete (hence uniformly complete and normal) f-algebra without unit element the converse of this proposition does not hold.

EXAMPLE 6.2. Let A be the Dedekind complete f-algebra of all real functions on $[0, 1]$ for which there exists $n \in \mathbf{N}$ such that $|f| \leqslant ni$. The r-ideal $I = \{ri + \alpha i: r \in A \text{ and } \alpha \in \mathbf{R}\}$ is not an l-ideal. Indeed, if $w(x) = \sqrt{x}$ for all $x \in [0, 1]$, then $wi \in A$, $0 \leqslant wi \leqslant i$, but $wi \notin I$.

On account of the present example it cannot be expected that the condition that every r-ideal in A is an l-ideal is equivalent to normality in a nonunitary f-algebra. We assume therefore in the remaining part of this section that the f-algebra A has a unit element.

In the next propositions we present some conditions on f-algebras which are equivalent to condition (γ) and condition (α) respectively.

PROPOSITION 6.3. *For an Archimedean f-algebra A with unit element e the following conditions are equivalent* (*compare* [**8**, *Theorem* 4.12]).

(i) *A is normal.*

(ii) *For any r-ideal I in A it follows from $f_1, \ldots, f_n \in I$, $f \in A$ and $(f - f_1) \cdots (f - f_n) = 0$ that $f \in I$.*

(iii) *For any $f \in A$ there exists $r \in A$ such that $f = r|f|$.*

(iv) $(f) = (|f|)$ *for all $f \in A$.*

(v) $(f, |f|)$ *is a principal r-ideal for all $f \in A$.*

PROOF. (i) $\Rightarrow$ (ii). Suppose $f \in A$, $f_1, \ldots, f_n \in I$ and $(f - f_1) \cdots (f - f_n) = 0$. Since A is semiprime we have $|f - f_1| \wedge \cdots \wedge |f - f_n| = 0$, and so it follows from the normality of A that $A = \{f - f_1\}^d + \cdots + \{f - f_n\}^d$ (see §2). Hence $e = e_1 + \cdots + e_n$, with $0 \leqslant e_i \in \{f - f_i\}^d$ $(i = 1, \ldots, n)$, i.e., $e_i(f - f_i) = 0$ $(i = 1, \ldots, n)$. This implies that $\sum_{i=1}^n e_i(f - f_i) = 0$, and therefore $f = \sum_{i=1}^n e_i f = \sum_{i=1}^n e_i f_i \in I$.

(ii) $\Rightarrow$ (iii). For any $f \in A$ we have $(f - |f|)(f + |f|) = 0$ and $|f| \in (|f|)$, so it follows from the hypothesis that $f \in (|f|)$, i.e. there exists $r \in A$ such that $f = r|f|$.

(iii) $\Rightarrow$ (iv). If we can prove that $|f| = rf$, then we are done. From $f = r|f|$ it follows that $f^+ - rf^+ = f^- + rf^-$, so $f^+ - rf^+ \perp f^- + rf^-$ implies $f^+ = rf^+$ and $f^- = -rf^-$. Hence $|f| = rf$.

(iv) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (i). Take $f \in A$ and suppose $(f, |f|) = (d)$. Hence there exist $g, h, s, t \in A$ such that $f = gd$, $|f| = hd$ and $d = sf + t|f|$. Using that $f^+ = \frac{1}{2}(h + g)d$, $f^- = \frac{1}{2}(h - g)d$ and that $(h^2 - g^2)d = 0$, we find

$$g - h \in \{f^+\}^d, \qquad g + h \in \{f^-\}^d. \tag{3}$$

Since $(sg + th - e)d = 0$, we have

$$(sg + th - e)f^+ = (sg + th - e)f^- = 0. \tag{4}$$

Denoting the l-ideal generated by an element $r \in A$ by $\langle r \rangle$ it is an easy matter to verify that $\langle r_1 \rangle + \langle r_2 \rangle = \langle |r_1| \vee |r_2| \rangle$ (see [**5**, 8.2.8]). Hence

$$\langle g + h \rangle + \langle g - h \rangle + \langle sg + th - e \rangle = \langle |g| \vee |h| \vee |sg + th - e| \rangle = \langle e \rangle = A.$$

Combining this with (3) and (4), we deduce that $A = \{f^+\}^d + \{f^-\}^d$.

REMARK 6.4. The equivalence of conditions (i) and (ii) in the above proposition was pointed out to us by W. A. J. Luxemburg.

PROPOSITION 6.5. *For an Archimedean f-algebra A with unit element the following conditions are equivalent.*

(i) *Every r-ideal in A is an l-ideal.*

(ii) *If* $0 \leqslant u \leqslant v$ *in* A, *then* $u = wv$ *for some* $0 \leqslant w \in A$.

(iii) $(f, g) = (|f| + |g|)$ *for all* $f, g \in A$.

(iv) $(f, g) = (|f| \vee |g|)$ *for all* $f, g \in A$.

PROOF. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Take $f, g \in A$. From $0 \leqslant f^+, f^- \leqslant |f|$ it follows that $f = r|f|$ for some $r \in A$; hence $|f| = rf$, and so $|f| \in (f, g)$. Similarly $|g| \in (f, g)$, and therefore $(|f| + |g|) \subset (f, g)$. Conversely, $|f| \leqslant |f| + |g|$ implies that $|f| = s(|f| + |g|)$ for some $s \in A$. Consequently, $f = rs(|f| + |g|)$, so $f \in (|f| + |g|)$. Analogously $g \in (|f| + |g|)$, which takes care of $(f, g) \subset (|f| + |g|)$.

(iii) $\Rightarrow$ (iv). Take $f, g \in A$. Now

$$(f, g) = (|f| + |g|) = (|f| \vee |g| + |f| \wedge |g|) = (|f| \vee |g|, |f| \wedge |g|).$$

Also $(|f| \vee |g|) = (|f| \wedge |g|, |f| \vee |g| - |f| \wedge |g|)$, and so $|f| \wedge |g| = t(|f| \vee |g|)$ for some $t \in A$. Combining these formulas we get $(f, g) = (|f| \vee |g|)$.

(iv) $\Rightarrow$ (i). Evident.

Now we shall prove the main theorem of this section.

THEOREM 6.6. *Consider the following conditions for the Archimedean f-algebra A with unit element e.*

(α) *Every r-ideal in A is an l-ideal.*

(β) *Every finitely generated r-ideal is a principal r-ideal.*

(γ) *A is normal.*

Then $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$. *Moreover, if A is uniformly complete, then* $(\alpha) \Leftrightarrow (\beta) \Leftrightarrow (\gamma)$.

PROOF. $(\alpha) \Rightarrow (\beta)$. If $I = (f_1, \ldots, f_n)$, then by Proposition 6.5, $I = (|f_1| + \cdots + |f_n|)$.

$(\beta) \Rightarrow (\gamma)$. From the hypothesis it follows that $(f, |f|)$ is a principal r-ideal for all $f \in A$. It follows from Proposition 6.3 that A is normal.

Suppose now that A is in addition uniformly complete.

$(\gamma) \Rightarrow (\alpha)$. By Proposition 6.5 it is sufficient to prove that $0 \leqslant u \leqslant v$ implies $u = wv$ for some $w \in A$. Define the positive orthomorphism $\pi_v \colon A \to A$ by $\pi_v f = vf$ for all $f \in A$. Since A is uniformly complete and normal, Theorem 5.2 states that R_{π_v} is an o-ideal. Since $v = \pi_v e \in R_{\pi_v}$, it follows that $u \in R_{\pi_v}$. Consequently, there exists $w \in A$ for which $u = \pi_v w = vw$.

Reformulating this theorem, we have that the range of every positive orthomorphism on a uniformly complete unitary f-algebra A is an o-ideal iff A is normal. From this it follows that it cannot be expected that Theorem 5.2 holds under weaker conditions.

We now present examples showing that $(\beta) \Rightarrow (\alpha)$ and $(\gamma) \Rightarrow (\beta)$ no longer hold in a nonuniformly complete f-algebra.

EXAMPLE 6.7. Let A be the f-algebra of all real sequences $f = (f(1), f(2), \ldots)$ for which there exist $n_f \in \mathbf{N}$ and $p_f \in \mathbf{R}[X]$ such that $f(n) = p_f(n)$ for all $n > n_f$. It is not hard to prove that A satisfies condition (β). However, the r-ideal $I = \{f \in A: p_f(1) = 0\}$ is not an l-ideal. Indeed, take $v(n) = n - 1$ ($n = 1, 2, \ldots$) and $u(n) = n - 2$ ($n = 2, 3, \ldots$), $u(1) = 0$. Then $0 \leqslant u \leqslant v \in I$, but $u \notin I$.

Note that A is normal, but the range of the orthomorphism π_v, defined by $\pi_v f = vf$, is not an o-ideal.

EXAMPLE 6.8. Let A be the f-algebra of all real functions f on $E = [0, 1] \times [0, 1]$ for which there exist disjoint sets $E_1, \ldots, E_{n_f}$ such that $E = \bigcup_{k=1}^{n_f} E_k$, and polynomials $p_k \in \mathbf{R}[X, Y]$, satisfying $f|E_k = p_k$ ($k = 1, \ldots, n_f$). Then A has the principal projection property, so A is normal. Using that the r-ideal (X, Y) in $\mathbf{R}[X, Y]$ is not principal, it is not difficult to show that the r-ideal (f, g), with $f(x, y) = x$ and $g(x, y) = y$, is not principal.

ADDED IN PROOF. Since any square-root closed Archimedean f-algebra is semiprime, the condition 'semiprime' in Theorem 4.12 can be omitted.

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